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Delay-dependent H_2 control for discrete time-delay systems with D-stability constraints

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Abstract

This paper studies the problem of H_2 control for a class of discrete time-delay systems with **D**-stability constraints. The corresponding sufficient conditions are given in terms of linear matrix inequalities. In particular, the conditions are delay-dependent, and so they are less conservative. The obtained controller can provide an upper bound for the H_2 cost function. A numerical example is given to illustrate the proposed method.

Keywords: Discrete systems; Time-delay; Linear matrix inequality; **D**-stability; H_2 control

1. Introduction

The problem of H_2 control for systems with **D**-stability constraints has attracted much attention [1–5]. For convenience of computation, the linear matrix inequality (LMI) method is then applied to solve this problem. For example, by enforcing a common matrix constraint, the $H_2/\mathbf{D}$-stable controller is designed [6,7]. Then, the results are developed in Refs. [8–10] with a parameter-dependent Lyapunov function. However, in the above-mentioned references, time-delay is not considered, while it is a source of instability in many cases. Therefore, the stability and performance analysis of time-delay systems is of theoretical and practical importance [11–13]. Furthermore, existing criteria for asymptotic stability of time-delay systems can be classified into two types: delay-independent stability and delay-dependent stability. And it is known that the latter is generally less conservative than the former especially when the size of the delay is small. In Ref. [14], the $H_2/\mathbf{D}$-stability controller synthesis problem is investigated for discrete time-delay systems, however, the H_2 control conditions are delay-independent.

In this paper, we obtain a delay-dependent solution to H_2 control problem for discrete systems with state delay and **D**-stability constraints. New H_2 performance specification with **D**-stability constraints is derived in terms of linear matrix inequalities (LMIs) that provide an upper bound for the H_2 cost function. And then, by minimizing this upper bound we derive the corresponding state-feedback controller.

2. Problem formulation and preliminaries

Consider the following discrete time-delay system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(k-d) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{B}_1\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \\ \boldsymbol{x}(k) &= \boldsymbol{\phi}(k), k \in [-d, 0] \end{aligned} \tag{1}$$

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where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state variable, $\boldsymbol{u}(k) \in \mathbb{R}^m$ is the control input, $\boldsymbol{z}(k) \in \mathbb{R}^p$ is the exogenous output, $\boldsymbol{w}(k) \in \mathbb{R}^q$ is the disturbance, d is a positive integer for delay time, $\boldsymbol{\phi}(k)$ is an initial value at k, and $\boldsymbol{A}$, $\boldsymbol{A}_{\mathrm{d}}$, $\boldsymbol{B}$, $\boldsymbol{B}_1$, $\boldsymbol{C}$, $\boldsymbol{D}$ are constant matrices of appropriate dimensions. Assume $\boldsymbol{x}(k)$ is measurable and consider the state-feedback $\boldsymbol{u}(k) = \boldsymbol{K}\boldsymbol{x}(k)$. Then the closed-loop system can be written as:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}_{\mathrm{cl}}\boldsymbol{x}(k) + \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(k-d) + \boldsymbol{B}_1\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}_{\mathrm{cl}}\boldsymbol{x}(k) \end{aligned} \tag{2}$$

where

$$\boldsymbol{A}_{\mathrm{cl}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}, \boldsymbol{C}_{\mathrm{cl}} = \boldsymbol{C} + \boldsymbol{D}\boldsymbol{K} \tag{3}$$

Also, we introduce H_2 performance measure as follows:

$$\boldsymbol{J}_{H_2} = \sum_{k=0}^{\infty} \boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k) \tag{4}$$

Our goal is to look for the feedback gain $\boldsymbol{K}$ such that the system (2) is asymptotically stable with **D**-stability constraints and the H_2 norm of the closed-loop transfer function $\boldsymbol{T}_{\boldsymbol{zw}}(s)$ from disturbance $\boldsymbol{w}$ to exogenous output $\boldsymbol{z}$ is minimized.

In what follows, for convenience, we use symbol $\mathbf{D}(0,1)$ to denote a unit circular region, symbol $\mathbf{D}(\alpha, r)$ a circular region with the center $(0, \alpha)$ and the radius r, where $|\alpha| < r < 1$, symbol $*$ the submatrix that lies below the diagonal.

Definition 1. System (1) is said to be **D**-stable if all eigenvalues of system matrix A lie in the region $\mathbf{D}(\alpha, r)$. Similarly, system (1) is called **D**-stabilizable if there exists a state-feedback $\boldsymbol{u}(k) = \boldsymbol{K}\boldsymbol{x}(k)$, such that the closed-loop system (2) is **D**-stable.

Lemma 1. *Given any real matrices $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ of appropriate dimensions and a symmetric and positive-definite matrix $\boldsymbol{G}_3$. Then, the following inequality holds*:

$$\boldsymbol{G}_1^{\mathrm{T}}\boldsymbol{G}_2 + \boldsymbol{G}_2^{\mathrm{T}}\boldsymbol{G}_1 \leqslant \boldsymbol{G}_1^{\mathrm{T}}\boldsymbol{G}_3\boldsymbol{G}_1 + \boldsymbol{G}_2^{\mathrm{T}}\boldsymbol{G}_3^{-1}\boldsymbol{G}_2$$

Lemma 2. [15] *Assume that $\boldsymbol{\alpha} \in \mathbb{R}^{n_{\mathrm{a}}}$, $\boldsymbol{\beta} \in \mathbb{R}^{n_{\mathrm{b}}}$ and $\boldsymbol{N} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{b}}}$. Then for any matrices $\boldsymbol{X} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{a}}}$, $\boldsymbol{Y} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{b}}}$ and $\boldsymbol{Z} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{b}}}$, the following inequality holds*:

$$-2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{\beta} \leqslant \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} - \boldsymbol{N} \\ * & \boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ * & \boldsymbol{Z} \end{bmatrix} \geqslant 0$$

3. Main results

3.1. *D-stability analysis and synthesis*

Let $\boldsymbol{u}(k) = 0, \boldsymbol{w}(k) = 0$, then the characteristic equation of system (1) is

$$\boldsymbol{\psi}(s) = det(zI - \boldsymbol{A} - \boldsymbol{A}_{\mathrm{d}}z^{-d}) = 0 \tag{5}$$

Now we state our first result as follows.

Theorem 1. *All the characteristic roots of system (1) are located inside $\mathbf{D}(\alpha, r)$ for all $d \in [0, \bar{d}]$, if there exist an integer $\bar{d} > 0$, symmetric and positive-definite matrices $\boldsymbol{X} > 0$ and $\boldsymbol{S} > 0$ such that*

$$\begin{bmatrix} (1,1) & (\boldsymbol{A} - \alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}} \\ * & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}} - \boldsymbol{S} \end{bmatrix} < 0 \tag{6}$$

where $(1,1) = (\boldsymbol{A} - \alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) - r^2\boldsymbol{X} + (r - |\alpha|)^{-\bar{d}}\boldsymbol{S}$.

Proof. Suppose there exists a characteristic root z of system (1) outside $\mathbf{D}(\alpha, r)$, then the inequality $(\bar{z} - \alpha)(z - \alpha) \geqslant r^2$ holds. For any symmetric and positive-definite matrix $\boldsymbol{X} > 0$, the inequality $(\bar{z} - \alpha)\boldsymbol{X}(z - \alpha) \geqslant r^2\boldsymbol{X}$ holds. Suppose $\boldsymbol{v} \in \mathbb{C}^n$ is the characteristic vector satisfying

$$z\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{A}_{\mathrm{d}}z^{-d}\boldsymbol{v} \tag{7}$$

Then multiplying both sides of the inequality by $\boldsymbol{v}^{\mathrm{H}}$ and $\boldsymbol{v}$ gives

$$\boldsymbol{v}^{\mathrm{H}}(\bar{z} - \alpha)\boldsymbol{X}(z - \alpha)\boldsymbol{v} \geqslant r^2\boldsymbol{v}^{\mathrm{H}}\boldsymbol{X}\boldsymbol{v} \tag{8}$$

Substituting (7) into (8) and rearranging the terms give

$$\begin{aligned} &\boldsymbol{v}^{\mathrm{H}}[(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) + (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}z^{-d} \\ &\quad + \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I})\bar{z}^{-d} + \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}\bar{z}^{-d}z^{-d} - r^2\boldsymbol{X}]\boldsymbol{v} \geqslant 0 \end{aligned} \tag{9}$$

From Lemma 1, it follows that for any symmetric and positive-definite matrix $\boldsymbol{S} > \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}$, the following inequality holds

$$\begin{aligned} &(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}z^{-d} + \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I})\bar{z}^{-d} \\ &\quad \leqslant (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) \\ &\quad + z^{-d}\bar{z}^{-d}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}) \end{aligned}$$

Again from (9), we can obtain

$$\begin{aligned} &\boldsymbol{v}^{\mathrm{H}}[(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) \\ &\quad + (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) + z^{-d}\bar{z}^{-d}\boldsymbol{S} - r^2\boldsymbol{X}]\boldsymbol{v} \geqslant 0 \end{aligned} \tag{10}$$

Note that $z^{-d}\bar{z}^{-d} \leqslant (r - |\alpha|)^{-d} \leqslant (r - |\alpha|)^{-\bar{d}}$ holds when $|\alpha| < r < 1$. Then (10) becomes

$$\begin{aligned} &\boldsymbol{v}^{\mathrm{H}}[(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) \\ &\quad - r^2\boldsymbol{X}. + (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) + (r - |\alpha|)^{-\bar{d}}\boldsymbol{S}]\boldsymbol{v} \geqslant 0 \end{aligned} \tag{11}$$

From Schur complement Lemma, (6) is equivalent to

$$\begin{aligned} &(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) \\ &\quad + (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) + (r - |\alpha|)^{-\bar{d}}\boldsymbol{S} - r^2\boldsymbol{X} < 0 \end{aligned}$$

Therefore, for any nonzero vector $\boldsymbol{w}$, there exist symmetric and positive-definite matrices $\boldsymbol{X} > 0$ and $\boldsymbol{S} > 0$ such that

$$\begin{aligned} &\boldsymbol{w}^{\mathrm{H}}[(\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{S} - \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}})^{-1}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) \\ &\quad + (\boldsymbol{A}^{\mathrm{T}} - \alpha\boldsymbol{I})\boldsymbol{X}(\boldsymbol{A} - \alpha\boldsymbol{I}) + (r - |\alpha|)^{-\bar{d}}\boldsymbol{S} - r^2\boldsymbol{X}]\boldsymbol{w} < 0 \end{aligned} \tag{12}$$

Obviously, (12) contradicts (11). This completes the proof. □

Corollary 1. *All the characteristic roots of system* (1) *are located inside* $\mathbf{D}(0,1)$ *if there exist symmetric and positive-definite matrices* $\boldsymbol{X}>0$ *and* $\boldsymbol{S}>0$ *such that*

$$\begin{bmatrix} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}-\boldsymbol{X}+\boldsymbol{S} & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}} \\ * & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{S} \end{bmatrix}<0 \tag{13}$$

Proof. The result can be obtained by taking $\alpha=0$ and $r=1$ in (6).

Since the term $(r-\mid\alpha\mid)^{d}$ nonlinearly appears in the LMI conditions and will cause troubles to apply the LMI toolbox, we have the following improved conclusion. □

Corollary 2. *All the characteristic roots of system* (1) *are located inside* $\mathbf{D}(\alpha,r)$ *for* $d\in[0,\bar{d}]$, *if there exist a positive scalar* $\lambda>1$, *symmetric and positive-definite matrices* $\boldsymbol{X}>0$ *and* $\boldsymbol{S}>0$ *such that*

$$\begin{bmatrix} (1,1) & (\boldsymbol{A}-\alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}} \\ * & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{S} \end{bmatrix}<0 \tag{14}$$

where

$$(1,1)=(\boldsymbol{A}-\alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{X}(\boldsymbol{A}-\alpha\boldsymbol{I})-r^{2}\boldsymbol{X}+\lambda\boldsymbol{S}$$
$$\bar{d}=\mathrm{int}\left[\frac{-\ln(\lambda)}{\ln(r-|\alpha|)}\right]$$

Proof. Substituting the positive scalar λ for $(r-\mid\alpha\mid)^{-\bar{d}}$ in Theorem 1 gives the conclusion of Corollary 2. □

Remark 1. The result of Corollary 2 is given in terms of LMIs with constraints, i.e. the generalized characteristic problem that can be solved by the Gevp solver in the LMI toolbox. When the minimum value of $-\lambda$ is obtained, the maximum value of the delay $\bar{d}$ can be reached by the equation $\lambda=(r-\mid\alpha\mid)^{-\bar{d}}$.

Now we design a state-feedback **D**-stability controller that locates all the closed-loop poles of system (2) inside $\mathbf{D}(\alpha,r)$.

Corollary 3. *There exists a state-feedback* $\boldsymbol{u}(k)=\boldsymbol{K}\boldsymbol{x}(k)$ *such that all the closed-loop poles of system* (2) *are located inside* $\mathbf{D}(\alpha,r)$ *for* $d\in[0,\bar{d}]$, *if there exist a positive scalar* $\lambda>1$, *symmetric and positive-definite matrices* $\boldsymbol{X}>0,\boldsymbol{S}>0$ *and matrix* $\boldsymbol{Y}$ *such that*

$$\begin{bmatrix} -r^{2}\boldsymbol{X}+\lambda\boldsymbol{S} & 0 & \boldsymbol{X}(\boldsymbol{A}-\alpha\boldsymbol{I})^{\mathrm{T}}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} \\ * & -\boldsymbol{S} & \boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} \\ * & * & -\boldsymbol{X} \end{bmatrix}<0 \tag{15}$$

where $\bar{d}=int[\frac{-ln(\lambda)}{ln(r-|\alpha|)}]$.

Furthermore, if the above condition holds, a desired state-feedback gain can be given by $\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{X}^{-1}$.

Proof. From Corollary 2, all the closed-loop poles of system (2) are located inside $\mathbf{D}(\alpha,r)$ for $d\in[0,\bar{d}]$, if there exist a positive scalar $\lambda>1$, symmetric and positive-definite matrices $\boldsymbol{P}>0$ and $\boldsymbol{Q}>0$ such that

$$\begin{bmatrix} (1,1) & (\boldsymbol{A}_{\mathrm{cl}}-\alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_{\mathrm{d}} \\ * & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{Q} \end{bmatrix}<0 \tag{16}$$

where $(1,1)=(\boldsymbol{A}_{\mathrm{cl}}-\alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}_{\mathrm{cl}}-\alpha\boldsymbol{I})-r^{2}\boldsymbol{P}+\lambda\boldsymbol{Q}$. From Schur complement Lemma, (16) is equivalent to

$$\begin{bmatrix} -r^{2}\boldsymbol{P}+\lambda\boldsymbol{Q} & 0 & (\boldsymbol{A}_{\mathrm{cl}}-\alpha\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P} \\ * & -\boldsymbol{Q} & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{P} \\ * & * & -\boldsymbol{P} \end{bmatrix}<0 \tag{17}$$

Pre- and post-multiply (17) by $diag\{\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1}\}$ and let $\boldsymbol{S}=\boldsymbol{P}^{-1}\boldsymbol{Q}\boldsymbol{P}^{-1},\boldsymbol{X}=\boldsymbol{P}^{-1}$, we can get

$$\begin{bmatrix} -r^{2}\boldsymbol{X}+\lambda\boldsymbol{S} & 0 & \boldsymbol{X}(\boldsymbol{A}_{\mathrm{cl}}-\alpha\boldsymbol{I})^{\mathrm{T}} \\ * & -\boldsymbol{S} & \boldsymbol{X}\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} \\ * & * & -\boldsymbol{X} \end{bmatrix}<0$$

Substitute $\boldsymbol{A}_{\mathrm{cl}}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$ into the above equation and let $\boldsymbol{Y}=\boldsymbol{K}\boldsymbol{X}$, then we can obtain (15). This completes the proof. □

3.2. H_2 synthesis for time-delay systems

In this section, we design an H_2 controller to stabilize the closed-loop system (2) and minimize the upper bound of H_2 cost of the system.

Theorem 2. *If there exist a positive scalar* $\gamma>0$, *symmetric and positive-definite matrices* $\boldsymbol{X}>0,\ \boldsymbol{T}_1>0,\ \boldsymbol{T}_2>0$ *and matrix* $\boldsymbol{T}_3$ *such that*

$$\begin{bmatrix} \boldsymbol{\Psi}_1 & 0 & \boldsymbol{B}_1 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_2 & \boldsymbol{\Psi}_3 & \boldsymbol{X} \\ * & -\boldsymbol{T}_2 & 0 & \boldsymbol{T}_2\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & \boldsymbol{T}_2\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & 0 & 0 \\ * & * & -\gamma\boldsymbol{I} & \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{B}_1^{\mathrm{T}} & 0 & 0 \\ * & * & * & -\boldsymbol{X} & 0 & 0 & 0 \\ * & * & * & * & -d^{-1}\boldsymbol{T}_1 & 0 & 0 \\ * & * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & * & -\boldsymbol{T}_2 \end{bmatrix}<0 \tag{18}$$

$$\begin{bmatrix} \boldsymbol{T}_3 & \boldsymbol{A}_{\mathrm{d}}\boldsymbol{T}_1 \\ * & \boldsymbol{T}_1 \end{bmatrix}\geqslant 0 \tag{19}$$

where $\boldsymbol{\Psi}_1=Sym\{(\boldsymbol{A}+\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{I})\boldsymbol{X}+\boldsymbol{B}\boldsymbol{Y}\}+d\boldsymbol{T}_3,\boldsymbol{\Psi}_2=\boldsymbol{X}\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}-\boldsymbol{X},\boldsymbol{\Psi}_3=\boldsymbol{X}\boldsymbol{C}^{\mathrm{T}}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}$, *then the control law* $\boldsymbol{u}(t)=\boldsymbol{K}\boldsymbol{x}(t)$, *where* $\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{X}^{-1}$, *stabilizes the closed-loop system* (2). *Furthermore, the corresponding* H_2 *cost* (4) *satisfies*

$$\boldsymbol{J}_{H_2} \leqslant \boldsymbol{J}_{H_2}^{*} = \boldsymbol{\phi}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{\phi}(0) + \sum_{l=-d}^{-1} \boldsymbol{x}^{\mathrm{T}}(l)\boldsymbol{S}_2\boldsymbol{x}(l) + \sum_{s=-d+1}^{0}\sum_{l=s-1}^{-1} \boldsymbol{y}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{y}(l) + \gamma\boldsymbol{I} \tag{20}$$

where $\boldsymbol{S}_1 = \boldsymbol{T}_1^{-1}, \boldsymbol{S}_2 = \boldsymbol{T}_2^{-1}$.

Proof. Let $\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + \boldsymbol{e}(k)$. Then

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{A}_{\mathrm{cl}}\boldsymbol{x}(k) + \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(k-d) + \boldsymbol{B}_1\boldsymbol{w}(k)\\ &= (\boldsymbol{A}_{\mathrm{cl}} + \boldsymbol{A}_{\mathrm{d}})\boldsymbol{x}(k) - \boldsymbol{A}_{\mathrm{d}}\sum_{l=k-d}^{k-1}\boldsymbol{e}(l) + \boldsymbol{B}_1\boldsymbol{w}(k)\end{aligned}$$

Consider a Lyapunov–Krasovskii functional candidate $\boldsymbol{V}(k)$ as follows:

$$\boldsymbol{V}(k) = \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{x}(k) + \sum_{l=1}^{d}\boldsymbol{x}^{\mathrm{T}}(k-l)\boldsymbol{S}_2\boldsymbol{x}(k-l) + \sum_{s=-d+1}^{0}\sum_{l=k-1+s}^{k-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l)$$

Then the forward difference of $\boldsymbol{V}(k)$ along the solution of system (2) is given by

$$\begin{aligned}\triangle\boldsymbol{V}(k) &= \boldsymbol{V}(k+1) - \boldsymbol{V}(k)\\ &= 2\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{e}(k) + \boldsymbol{e}^{\mathrm{T}}(k)(\boldsymbol{P}+d\boldsymbol{S}_1)\boldsymbol{e}(k)\\ &\quad - \sum_{l=k-d}^{k-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l) + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{x}(k) - \boldsymbol{x}^{\mathrm{T}}(k-d)\boldsymbol{S}_2\boldsymbol{x}(k-d)\\ &= 2\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}(\boldsymbol{A}_{\mathrm{cl}}+\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{I})\boldsymbol{x}(k) + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{x}(k) + \boldsymbol{e}^{\mathrm{T}}(k)(\boldsymbol{P}+d\boldsymbol{S}_1)\boldsymbol{e}(k)\\ &\quad - 2\sum_{l=k-d}^{k-1}\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{A}_{\mathrm{d}}\boldsymbol{e}(l) + 2\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{w}(k) - \boldsymbol{x}^{\mathrm{T}}(k-d)\boldsymbol{S}_2\boldsymbol{x}(k-d)\\ &\quad - \sum_{l=k-d}^{k-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l)\end{aligned} \tag{21}$$

By Lemma 2, we can get

$$\begin{aligned}&-2\sum_{l=k-d}^{k-1}\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{A}_{\mathrm{d}}\boldsymbol{e}(l)\\ &\leqslant \sum_{l=k-d}^{k-1}\begin{bmatrix}\boldsymbol{x}^{\mathrm{T}}(k) & \boldsymbol{e}^{\mathrm{T}}(l)\end{bmatrix}\begin{bmatrix}\boldsymbol{S}_3 & 0\\ * & \boldsymbol{S}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{e}(l)\end{bmatrix}\\ &= d\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{x}(k) + \sum_{l=k-d}^{k-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l)\end{aligned} \tag{22}$$

where $\boldsymbol{S}_1, \boldsymbol{S}_3$ are constant matrices of appropriate dimensions satisfying

$$\begin{bmatrix}\boldsymbol{S}_3 & \boldsymbol{P}\boldsymbol{A}_{\mathrm{d}}\\ * & \boldsymbol{S}_1\end{bmatrix} \geqslant 0 \tag{23}$$

From (21) to (22), we can get

$$\triangle\boldsymbol{V}(k) \leqslant \boldsymbol{\xi}^{\mathrm{T}}(k)\boldsymbol{\Xi}\boldsymbol{\xi}(k) - \boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k) + \gamma\boldsymbol{w}^{\mathrm{T}}(k)\boldsymbol{w}(k) \tag{24}$$

where

$$\begin{aligned}\boldsymbol{\xi}^{\mathrm{T}}(k) &= \begin{bmatrix}\boldsymbol{x}^{\mathrm{T}}(k) & \boldsymbol{x}^{\mathrm{T}}(k-d) & \boldsymbol{w}^{\mathrm{T}}(k)\end{bmatrix}\\ \boldsymbol{\Xi} &= \begin{bmatrix}(1,1) & 0 & \boldsymbol{P}\boldsymbol{B}_1\\ * & -\boldsymbol{S}_2 & 0\\ * & * & -\gamma\boldsymbol{I}\end{bmatrix} + \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{P}+d\boldsymbol{S}_1)\boldsymbol{\varphi}\\ \boldsymbol{\varphi} &= \begin{bmatrix}\boldsymbol{A}_{\mathrm{cl}}-\boldsymbol{I} & \boldsymbol{A}_{\mathrm{d}} & \boldsymbol{B}_1\end{bmatrix}\\ (1,1) &= Sym\{\boldsymbol{P}(\boldsymbol{A}_{\mathrm{cl}}+\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{I})\} + d\boldsymbol{S}_3 + \boldsymbol{S}_2 + \boldsymbol{C}_{\mathrm{cl}}^{\mathrm{T}}\boldsymbol{C}_{\mathrm{cl}}\end{aligned}$$

If there exist symmetric and positive-definite matrices $\boldsymbol{P} > 0, \boldsymbol{S}_1 > 0, \boldsymbol{S}_2 > 0$ and matrix $\boldsymbol{S}_3$ such that $\boldsymbol{\Xi} < 0$, then we can get

$$\triangle\boldsymbol{V}(k) < -\boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k) + \gamma\boldsymbol{w}^{\mathrm{T}}(k)\boldsymbol{w}(k) \tag{25}$$

Obviously, (25) ensures the asymptotical stability of the closed-loop system (2). Furthermore, summing both sides of (25) from zero to ∞ and using the initial conditions yield

$$\sum_{k=0}^{\infty}\boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k) \leqslant \boldsymbol{\phi}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{\phi}(0) + \sum_{l=-d}^{-1}\boldsymbol{x}^{\mathrm{T}}(l)\boldsymbol{S}_2\boldsymbol{x}(l) + \sum_{s=-d+1}^{0}\sum_{l=s-1}^{-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l) + \gamma\boldsymbol{I} = \boldsymbol{J}_{H_2}^{*} \tag{26}$$

Thus, if there exist symmetric and positive-definite matrices $\boldsymbol{P} > 0, \boldsymbol{S}_1 > 0, \boldsymbol{S}_2 > 0$ and matrix $\boldsymbol{S}_3$ such that the equation $\boldsymbol{\Xi} < 0$ and (23) holds, system (2) is stable and the corresponding H_2 cost (4) satisfies $\boldsymbol{J}_{H_2} \leqslant \boldsymbol{J}_{H_2}^{*}$. By Schur complement Lemma, the equation $\boldsymbol{\Xi} < 0$ is equivalent to the following equation

$$\begin{bmatrix}(1,1) & 0 & \boldsymbol{P}\boldsymbol{B}_1 & \boldsymbol{A}_{\mathrm{cl}}^{\mathrm{T}}-\boldsymbol{I} & (1,5) & \boldsymbol{C}_{\mathrm{cl}}^{\mathrm{T}} & \boldsymbol{I}\\ * & -\boldsymbol{S}_2 & 0 & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & \boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & 0 & 0\\ * & * & -\gamma\boldsymbol{I} & \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{B}_1^{\mathrm{T}} & 0 & 0\\ * & * & * & -\boldsymbol{P}^{-1} & 0 & 0 & 0\\ * & * & * & * & (5,5) & 0 & 0\\ * & * & * & * & * & -\boldsymbol{I} & 0\\ * & * & * & * & * & * & -\boldsymbol{S}_2^{-1}\end{bmatrix} < 0 \tag{27}$$

where $(1,1) = Sym\{\boldsymbol{P}(\boldsymbol{A}_{\mathrm{cl}}+\boldsymbol{A}_{\mathrm{d}}-\boldsymbol{I})\} + d\boldsymbol{S}_3, (1,5) = \boldsymbol{A}_{\mathrm{cl}}^{\mathrm{T}} - \boldsymbol{I}, (5,5) = -d^{-1}\boldsymbol{S}_1^{-1}$. Let $\boldsymbol{X} = \boldsymbol{P}^{-1}, \boldsymbol{T}_1 = \boldsymbol{S}_1^{-1}, \boldsymbol{T}_2 = \boldsymbol{S}_2^{-1}, \boldsymbol{T}_3 = \boldsymbol{X}\boldsymbol{S}_3\boldsymbol{X}$, make a congruence transformation on (27) with $diag\{\boldsymbol{X}, \boldsymbol{T}_2, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}\}$ and make a congruence transformation on (23) with $diag\{\boldsymbol{X}, \boldsymbol{T}_1\}$, then we can get

$$\begin{bmatrix}(1,1) & 0 & \boldsymbol{B}_1 & (1,4) & (1,5) & \boldsymbol{X}\boldsymbol{C}_{\mathrm{cl}}^{\mathrm{T}} & \boldsymbol{X}\\ * & -\boldsymbol{T}_2 & 0 & \boldsymbol{T}_2\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & \boldsymbol{T}_2\boldsymbol{A}_{\mathrm{d}}^{\mathrm{T}} & 0 & 0\\ * & * & -\gamma\boldsymbol{I} & \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{B}_1^{\mathrm{T}} & 0 & 0\\ * & * & * & -\boldsymbol{X} & 0 & 0 & 0\\ * & * & * & * & -d^{-1}\boldsymbol{T}_1 & 0 & 0\\ * & * & * & * & * & -\boldsymbol{I} & 0\\ * & * & * & * & * & * & -\boldsymbol{T}_2\end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix}\boldsymbol{T}_3 & \boldsymbol{A}_{\mathrm{d}}\boldsymbol{T}_1\\ * & \boldsymbol{T}_1\end{bmatrix} \geqslant 0 \tag{29}$$

where $(1,1) = Sym\{(\boldsymbol{A}_{\mathrm{cl}} + \boldsymbol{A}_{\mathrm{d}} - \boldsymbol{I})\boldsymbol{X}\} + d\boldsymbol{T}_3, (1,4) = (1,5) = \boldsymbol{X}\boldsymbol{A}_{\mathrm{cl}}^{\mathrm{T}} - \boldsymbol{X}$. Substituting (3) to (28) and letting $\boldsymbol{Y} = \boldsymbol{K}\boldsymbol{X}$, we can get (18). This completes the proof. □

3.3. $H_2/\mathbf{D}$-stability controller design

Theorem 3. *Consider the discrete time-delay system* (2). *If the following optimization problem*

$$\min\{\alpha + tr(\boldsymbol{Q}_1) + tr(\boldsymbol{Q}_2) + \gamma\} \tag{30}$$

subject to

(*i*) (15), (18) *and* (19)

(*ii*) $\begin{bmatrix} -\alpha & \boldsymbol{\phi}^{\mathrm{T}}(0) \\ * & -\boldsymbol{X} \end{bmatrix} < 0$

(*iii*) $\begin{bmatrix} -\boldsymbol{Q}_1 & \boldsymbol{N}^{\mathrm{T}} \\ * & -\boldsymbol{T}_2 \end{bmatrix} < 0$

(*iv*) $\begin{bmatrix} -\boldsymbol{Q}_2 & \boldsymbol{M}^{\mathrm{T}} \\ * & -\boldsymbol{T}_1 \end{bmatrix} < 0$

has a solution with a positive scalar α, *symmetric and positive matrices* $\boldsymbol{X}, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{Q}_1, \boldsymbol{Q}_2$ *and matrices* $\boldsymbol{Y}, \boldsymbol{T}_3$, *then system* (2) *is stable and the corresponding* H_2 *cost satisfies*

$$\boldsymbol{J}_{H_2} \leqslant \boldsymbol{J}_{H_2}^* < \boldsymbol{J}^* = \alpha + \mathrm{tr}(\boldsymbol{Q}_1) + tr(\boldsymbol{Q}_2) + \gamma \tag{31}$$

And $\boldsymbol{u}(t) = \boldsymbol{Y}\boldsymbol{X}^{-1}$ *is an* $H_2/\mathbf{D}$*-stable controller of the system* (2). *Here, tr* $(\cdot)$ *denotes the trace of the matrix* $(\cdot)$ *and* $\sum_{i=1}^{d}\boldsymbol{\phi}(-i)\boldsymbol{\phi}^{\mathrm{T}}(-i) = \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}, \sum_{s=-d+1}^{0}\sum_{l=s-1}^{-1}\boldsymbol{e}(l)\boldsymbol{e}^{\mathrm{T}}(l) = \boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}$.

Proof. Let $\boldsymbol{X} = \boldsymbol{P}^{-1}, \boldsymbol{S}_2 = \boldsymbol{T}_2^{-1}, \boldsymbol{S}_1 = \boldsymbol{T}_1^{-1}$. (ii) is equivalent to $\boldsymbol{\phi}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{\phi}(0) < \alpha$. (iii) and (iv) are, respectively, equivalent to the following equations

$$-\boldsymbol{Q}_1 + \boldsymbol{N}^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{N} < 0, -\boldsymbol{Q}_2 + \boldsymbol{M}^{\mathrm{T}}\boldsymbol{S}_1\boldsymbol{M} < 0 \tag{32}$$

From (20), we can get

$$\begin{aligned} \sum_{i=1}^{d}\boldsymbol{\phi}^{\mathrm{T}}(-i)\boldsymbol{S}_2\boldsymbol{\phi}(-i) &= \sum_{i=1}^{d}\mathrm{tr}(\boldsymbol{\phi}^{\mathrm{T}}(-i)\boldsymbol{S}_2\boldsymbol{\phi}(-i)) = \mathrm{tr}(\boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{S}_2) \\ &= \mathrm{tr}(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{N}) < \mathrm{tr}(\boldsymbol{Q}_1) \end{aligned} \tag{33}$$

$$\begin{aligned} \sum_{s=-d+1}^{0}\sum_{l=s-1}^{-1}\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l) &= \sum_{s=-d+1}^{0}\sum_{l=s-1}^{-1}\mathrm{tr}(\boldsymbol{e}^{\mathrm{T}}(l)\boldsymbol{S}_1\boldsymbol{e}(l)) \\ &= \mathrm{tr}(\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{S}_1) \\ &= \mathrm{tr}(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{S}_1\boldsymbol{M}) < \mathrm{tr}(\boldsymbol{Q}_2) \end{aligned} \tag{34}$$

Then we can get

$$\boldsymbol{J}_{H_2}^* < \alpha + \mathrm{tr}(\boldsymbol{Q}_1) + \mathrm{tr}(\boldsymbol{Q}_2) + \gamma = \boldsymbol{J}^* \tag{35}$$

This completes the proof. □

Remark 2. The generalized characteristic problem (15) and the optimization problem (30) cannot be solved by the LMI toolbox simultaneously. Therefore, the problem (15) can be firstly solved to find the maximum value of the time delay d. Then the problem (30) can be solved by the LMI toolbox after the time-delay d is made certain.

4. Illustrative example

Consider the following discrete time-delay system

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_{\mathrm{d}}\boldsymbol{x}(k-d) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{B}_1\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \end{aligned} \tag{36}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1.2 \end{bmatrix}, \boldsymbol{A}_{\mathrm{d}} = \begin{bmatrix} -0.25 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\boldsymbol{B}_1 = [0.1 \quad 0.1]^{\mathrm{T}}, \boldsymbol{C} = [0.3 \quad 0.3], \boldsymbol{D} = 4$$

$$\boldsymbol{\phi}(k) = [\mathrm{e}^{-k} \quad 0]^{\mathrm{T}}$$

Let $\boldsymbol{u}(k) = 0$. It is expected to design a state-feedback **D**-stability controller locating all the closed-loop poles inside $\mathbf{D}(\alpha, r)$. Based on Corollary 3, we can find the maximum time delay $\bar{d}$ and its corresponding state-feedback gain $\boldsymbol{K}$ of system (36) with different α and r. We obtain $\bar{d} = 3$, $\boldsymbol{K} = [-0.0292 \quad -0.1948]$ when $\alpha = -0.2, r = 0.8$; $\bar{d} = 2$, $\boldsymbol{K} = [0.0058 \quad -0.0720]$ when $\alpha = 0.1, r = 0.6$; $\bar{d} = 1$, $\boldsymbol{K} = [0 \quad -0.1132]$ when $\alpha = 0, r = 0.5$.

We can see that the value of the maximum time-delay $\bar{d}$ is smaller when the range of the specified region is smaller. And the state-feedback gain $\boldsymbol{K} = [-0.0292 \quad -0.1948]$ locates all the closed-loop poles of the corresponding system without delay inside $\mathbf{D}(0, 1)$. Therefore, time-delay increases the conservativeness of the system.

Then let $\boldsymbol{u}(k) = \boldsymbol{K}\boldsymbol{x}(t)$. We design a state-feedback H_2/**D**-stability controller that not only guarantees all the closed-loop poles to be placed inside a specified region $\mathbf{D}(\alpha, r)$ ($\alpha = 0.2, r = 0.6$), but also provides an H_2 performance upper bound. Given $d = 1$, from Theorem 3, we obtain the H_2 cost bound $\boldsymbol{J}^* = 1.5152$ of disturbance attenuation level and its corresponding state-feedback gain $\boldsymbol{K} = [0.0031 \quad -0.0680]$.

5. Conclusions

The H_2 control problem has been considered for a class of discrete time-delay systems with **D**-stability constraints. The obtained **D**-stability criterion and H_2 performance specification are delay-dependent and given in terms of LMIs. Based on the above conditions, a mixed H_2/**D**-stability controller is designed that not only guarantees all the closed-loop poles are placed inside a specified region, but also provides an H_2 performance upper bound. Further results on H_2/**D**-stability control for uncertain discrete time systems with time-delay will be presented elsewhere.

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